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A Note on Equivalent Comparisons of Information Channels*

Luís Fernando Brands Barbosa[†] Gil Riella[‡]

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Abstract

Nakata (2011) presents a model of acquisition of information where the agent does not know what pieces of information she is missing. In this note we point out some technical problems in a few of Nakata's results and show how to correct them.

Keywords: Information channels, Preferences over Menus, Weak Expected Utility Representation, Ordinal Expected Utility Representation, Additive Expected Utility Representation.

1 Introduction

Nakata (2011) presents a model of acquisition of information where the agent does not know what pieces of information she is missing. His results are heavilly based on Dekel, Lipman, and Rustichini (2001)—henceforth DLR. In this note we point out some problems in a few of his results and show how to correct them.

In the next section we present the formal setup and introduce the main representation concepts we are going to work with. In Section 3, we discuss the problem in Nakata's Theorem 1. We observe that the continuity axiom used by Nakata is not strong enough to imply the desired representation. In fact, we give an example that shows that Nakata's axioms are not

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sufficient for the relation to have a representation by a utility function. Later we show how to strenghen Nakata's continuity axiom in order to obtain the desired representation.

In Section 4, we argue that Nakata's Theorem 2 also suffers from the same problem as his first theorem. This problem can be fixed the same way we fixed the first result.

In Section 5, we discuss Nakata's Theorem 3, which deals with Monotone Additive EU representations. Again, this result suffers from the same problem we have pointed out above. However, in this case it is not enough to strengthen the continuity axiom. We show through an example that even if we strengthen Nakata's continuity axiom the same way we did before we still do not obtain the desired representation. We then present an additional postulate and show that when we add it to the other axioms we obtain the correct version of Nakata's Theorem 3.

Finally, in section 6, we introduce an additional postulate that guarantees that the sets of information nodes in Theorems 2 and 3 are all finite.

2 Setup and Representations

Let's denote by $\mathcal{J}$ the finite set of information channels. Let C be a finite set of outcomes and $\Delta(C)$ be the set of all probability distributions (lotteries) on C, which we view as a metric subspace of $\mathbb{R}^n$. Denote by $\mathcal{B}(\Delta(C))$ the class of all nonempty closed subsets of $\Delta(C)$, which we view as a metric space under the Hausdorff metric.[1] We refer to the the elements of $\mathcal{B}(\Delta(C))$ as menus and note that the space of menus is compact. The agent has a preference relation (a complete and transitive binary relation) $\succsim$ on $\mathcal{J} \times \mathcal{B}(\Delta(C))$. As usual, we denote the symmetric part of $\succsim$ by $\sim$ and the asymmetric part by $\succ$. We will work with the following definition:

Definition 1. A *Weak EU representation* is a tuple $(\{\mathcal{M}_j, \varphi_j\}_{j\in\mathcal{J}}, U)$, such that

i For each $j \in \mathcal{J}$, $\mathcal{M}_j$ is any set;

ii $U : \Delta(C) \times \cup_{j\in\mathcal{J}}\mathcal{M}_j \to \mathbb{R}$ is such that, for each $j \in \mathcal{J}$ and each $m_j \in \mathcal{M}_j$, $U(., m_j)$ is a nontrivial expected utility function;[2]

[1] Nakata (2011) follows DLR and defines $\mathcal{B}(\Delta(C))$ to be the class of all subsets of $\Delta(C)$. We restrict ourselves to closed subsets of $\Delta(C)$ to simplify the exposition.

[2] That is, there exists a non-constant function $u : C \to \mathbb{R}$ such that, for every $p \in \Delta(C)$, $U(p, m_j) = \sum_{x\in C} p(x)u(x)$.

iii The functions $\varphi_j : \mathbb{R}^{\mathcal{M}_j} \to \mathbb{R}$ are such that the function $v : \mathcal{J} \times \mathcal{B}(\Delta(C)) \to \mathbb{R}$ defined by

$$v(j, A) := \varphi_j \left(\left\langle \max_{p \in A} U(p, m_j) \right\rangle_{m_j \in \mathcal{M}_j} \right),$$

for every $j \in \mathcal{J}$ and every $A \in \mathcal{B}(\Delta(C))$, is continuous.

If the sets $\{\mathcal{M}_j\}_{j \in \mathcal{J}}$ all have a finite number of elements, then we say that $(\{\mathcal{M}_j, \varphi_j\}_{j \in \mathcal{J}}, U)$ is a *Finite Weak EU representation.*

Nakata puts two additional conditions on the definition of a Weak EU representation, but they concern only the existence of irrelevant and redundant elements in the sets $\{\mathcal{M}_j\}_{j \in \mathcal{J}}$ and are not important for the analysis here.

We say that a Weak EU representation $(\{\mathcal{M}_j, \varphi_j\}_{j \in \mathcal{J}}, U)$ represents a preference relation $\succsim$ on $\mathcal{J} \times \mathcal{B}(\Delta(C))$ if the function v as defined in statement (iii) of definition 1 represents it.[3]

We will also need the following definitions:

Definition 2. An *Ordinal EU representation* is a Weak EU representation $(\{\mathcal{M}_j, \varphi_j\}_{j \in \mathcal{J}}, U)$ where the aggregators $\{\varphi_j\}_{j \in \mathcal{J}}$ are all strictly increasing on their relevant domain.[4] Again, if all the sets $\{\mathcal{M}_j\}_{j \in \mathcal{J}}$ are finite, we say that $(\{\mathcal{M}_j, \varphi_j\}_{j \in \mathcal{J}}, U)$ is a *Finite Ordinal EU representation.*

Definition 3. A *Monotone Additive EU representation* is a tuple $(\{(\mathcal{M}_j, \Sigma_j), \mu_j\}_{j \in \mathcal{J}}, U)$ such that

i For each $j \in \mathcal{J}$, $(\mathcal{M}_j, \Sigma_j)$ is a measurable space;

ii For each $j \in \mathcal{J}$, μ_j is a finite positive measure on $(\mathcal{M}_j, \Sigma_j)$;

iii $U : \Delta(C) \times \cup_{j \in \mathcal{J}} \mathcal{M}_j \to \mathbb{R}$ is a measurable state dependent function such that, for each $j \in \mathcal{J}$ and each $m_j \in \mathcal{M}_j$, $U(., m_j)$ is a nontrivial expected utility function.

We say that $(\{(\mathcal{M}_j, \Sigma_j), \mu_j\}_{j \in \mathcal{J}}, U)$ represents $\succsim$ if, for every (i, A) and (j, B) in $\mathcal{J} \times \mathcal{B}(\Delta(C))$, $(i, A) \succsim (j, B)$ iff

$$\int_{\mathcal{M}_i} \max_{p \in A} U(p, m_i) \mu_i(dm_i) \geq \int_{\mathcal{M}_j} \max_{p \in B} U(p, m_j) \mu_j(dm_j).$$

[3] That is, if, for every $i, j \in \mathcal{J}$ and $A, B \in \mathcal{B}(\Delta(C))$, we have $(i, A) \succsim (j, B)$ iff $v(i, A) \geq v(j, B)$.

[4] That is, every φ_j is strictly increasing on $\{\langle \max_{p \in A} U(p, m_j) \rangle_{m_j \in \mathcal{M}_j} : A \in \mathcal{B}(\Delta(C))\}$.

Again, if all the sets $\{\mathcal{M}_j\}_{j\in\mathcal{J}}$ are finite, we say that $(\{(\mathcal{M}_j,\Sigma_j),\mu_j\}_{j\in\mathcal{J}},U)$ is a *Finite Monotone Additive EU representation.*

3 Characterization of Weak EU Representations

Theorem 1 in Nakata (2011) claims that Weak EU representations (with no irrelevant nodes) are characterized by the following postulates:

Axiom 1 (Continuity). *For every $j\in\mathcal{J}$ and $A\in\mathcal{B}(\Delta(C))$, the sets $\{(j,B):B\in\mathcal{B}(\Delta(C))$ and $(j,A)\succ(j,B)\}$ and $\{(j,B):B\in\mathcal{B}(\Delta(C))$ and $(j,B)\succ(j,A)\}$ are open.*

Axiom 2 (Nontriviality). *For every $j\in\mathcal{J}$, there exist some A and B in $\mathcal{B}(\Delta(C))$ such that $(j,A)\succ(j,B)$.*

Axiom 3 (Indifference to Randomization). *For every $j\in\mathcal{J}$ and $A\in\mathcal{B}(\Delta(C))$, $(j,A)\sim(j,conv(A))$, where $conv(A)$ denotes the convex hull of A.*

However, it turns out that the Continuity axiom above is too weak to guarantee that $\succsim$ admits a Weak EU representation. In fact, in the example below we show that a particular type of lexicographic relation satisfies all the axioms above. We then use a standard argument to show that this relation does not admit a representation by a utility function.

Example 1. Let $\mathcal{J}:=\{j,j^{'}\}$, $C:=\{x,y\}$ and $u:C\longrightarrow\mathbb{R}$ be given by $u(x):=1,u(y):=0$. Define a preference relation $\succsim$ on $\mathcal{J}\times\mathcal{B}(\Delta(C))$ by

$$(k,A)\succ(l,B)\ \text{ iff } \max_{p\in A}\mathbb{E}_p(u)>\max_{p\in B}\mathbb{E}_p(u)\ \text{ or } \begin{cases}\max_{p\in A}\mathbb{E}_p(u)=\max_{p\in B}\mathbb{E}_p(u)\\ \text{and}\\ k=j \text{ and } l=j^{'}\end{cases}.^{5} \tag{1}$$

We can easily check that $\succsim$ satisfies all the axioms above. However, we can use a standard argument to show that it does not admit a representation by a utility function. To see that, suppose that there exists a function $W:\mathcal{J}\times\mathcal{B}(\Delta(C))\longrightarrow\mathbb{R}$ such that, for each $(k,A),(l,B)\in\mathcal{J}\times\mathcal{B}(\Delta(C))$ we have

$$W(k,A)\geq W(l,B)\Longleftrightarrow(k,A)\succsim(l,B).$$

This implies that, for each $\lambda\in[0,1]$, we have $W(j,\{\lambda x\oplus(1-\lambda)y\})>W(j^{'},\{\lambda x\oplus(1-\lambda)y\})$ and, therefore, we can pick a rational number $q_\lambda\in(W(j^{'},\{\lambda x\oplus(1-\lambda)y\}),W(j,\{\lambda x\oplus$

[5] *Notation.* For any lottery $p\in\Delta(C)$, $\mathbb{E}_p(u):=p(x)u(x)+p(y)u(y)$.

$(1-\lambda)\, y\}))$.[6] Since W represents $\succsim$, for each $\lambda, \lambda^{'} \in [0,1]$ such that $\lambda > \lambda^{'}$ we have $q_{\lambda} > W(j^{'}, \{\lambda x \oplus (1-\lambda) y\}) > W(j, \{\lambda' x \oplus (1-\lambda') y\}) > q_{\lambda'}$. But then, $q_{()}$ is an injective function from $[0,1]$ into $\mathbb{Q}$, which is a contradiction. We conclude that $\succsim$ is not representable by a utility function. ‖

To avoid situations such as the one described in the previous example, we need to strengthen the Continuity axiom above.

Axiom 4 (Continuity II). *For every $j \in \mathcal{J}$ and $A \in \mathcal{B}(\Delta(C))$, the sets $\{(j', B) \in \mathcal{J} \times \mathcal{B}(\Delta(C)) : (j', B) \succ (j, A)\}$ and $\{(j', B) \in \mathcal{J} \times \mathcal{B}(\Delta(C)) : (j, A) \succ (j', B)\}$ are open.*

We can now state the following result:

Theorem 1. *A preference relation $\succsim$ satisfies Axioms 3 and 4 if, and only if, it admits a Weak EU representation.*[7]

4 Characterization of Ordinal EU Representations

Consider now the following postulates.

Axiom 5 (Weak Independence). *If $A \subseteq B$, then for all $\lambda \in (0,1]$, $\bar{B} \in \mathcal{B}(\Delta(C))$ and $j \in \mathcal{J}$,*

$$\begin{aligned}(j, B) \;\succ\; (j, A) &\implies (j, \lambda B + (1-\lambda)\bar{B}) \succ (j, \lambda A + (1-\lambda)\bar{B}),\\ (j, B) \;\sim\; (j, A) &\implies (j, \lambda B + (1-\lambda)\bar{B}) \sim (j, \lambda A + (1-\lambda)\bar{B}).\end{aligned}$$[8]

Axiom 6 (Monotonicity). *If $B \subseteq A$, then $(j, A) \succsim (j, B)$ for all $j \in \mathcal{J}$.*

Nakata's Theorem 2 claims that Axioms 1, 2, 5 and 6 characterize the preference relations that admit an Ordinal EU representation. This claim suffers from the same problem as his first result and, in particular, Example 1 satisfies all the axioms above. Again, the way to remedy this situation is to strengthen Continuity to Continuity II.

[6] *Notation.* By $\lambda x \oplus (1-\lambda) y$ we mean the lottery that pays prize x with probability λ and prize y with probability $1-\lambda$.

[7] Since we did not include the requirement that all the information nodes in the representation be relevant, we cannot exclude the possibility that $(j, A) \sim (j, B)$ for all A and B in $\mathcal{B}(\Delta(C))$, for some $j \in \mathcal{J}$. This is the reason why Axiom 2 is not part of the postulates in Theorem 1.

[8] For any pair of menus A and B, and any $\lambda \in [0,1]$, the convex combination of A and B is defined by $\lambda A + (1-\lambda) B := \{\lambda p + (1-\lambda) q : p \in A \text{ and } q \in B\}$.

That is, we can prove the following result:

Theorem 2. *A preference relation* $\succsim$ *satisfies Axioms 2, 4, 5 and 6 if, and only if, it admits an Ordinal EU representation.*[9]

5 Characterization of Monotone Additive Eu Representations

Consider now the following postulate.

Axiom 7 (Independence)**.** *If* $(j, A) \succ (j, B)$*, then for all* $\lambda \in (0, 1]$ *and* $\bar{B} \in \mathcal{B}(\Delta(C))$ *we have*

$$(j, \lambda A + (1 - \lambda)\bar{B}) \succ (j, \lambda B + (1 - \lambda)\bar{B}).$$

Nakata's Theorem 3 claims that Axioms 1, 2, 6 and 7 characterize the preference relations that admit a Monotone Additive EU representation. Once again, this claim suffers from the problem pointed out in the previous sections. That is, they do not ensure that $\succsim$ can be represented by a utility function. Notice that Example 1 satisfies all the axioms above.

As we did in the previous section, this can be fixed by replacing Axiom 1 by Axiom 4. However, example 2 below shows this is not enough to guarantee that $\succsim$ admits a Monotone Additive EU representation.

It turns out that the postulates discussed above are only sufficient to guarantee that, once we fix a particular information channel $j \in \mathcal{J}$, the restriction of $\succsim$ to $\{j\} \times \mathcal{B}(\Delta(C))$ admits a Monotone Additive EU representation. In order to guarantee that the entire relation $\succsim$ admits such a representation, we need a property that ties the Monotone Additive EU representations of the different information channels together. For example, consider the following postulate.

Axiom 8 (Separability)**.** *For any* $j, k \in \mathcal{J}$*,* $A, B, A', B' \in \mathcal{B}(\Delta(C))$ *and* $\lambda \in (0, 1)$*, if* $(j, A) \sim (k, B)$ *and* $(j, A') \sim (k, B')$*, then* $(j, \lambda A + (1 - \lambda) A') \sim (k, \lambda B + (1 - \lambda) B')$*.*

It is easily checked that if $\succsim$ admits a Monotone Additive EU representation, then $\succsim$ satisfies Axiom 8. However, Axioms 2, 4, 6 and 7 do not imply Axiom 8. Consider the following example:

[9] Now, the requirement that all the aggregators $\{\varphi_j\}_{j \in \mathcal{J}}$ are strictly increasing implies that Axiom 2 is satisfied. This is the reason why this postulate is present in the statement of Theorem 2.

Example 2. Let $\mathcal{J} := \{j, k\}$, $C := \{x, y\}$. Define

$$u_j(x) := 9; u_j(y) := 1;$$
$$u_k(x) := 1; u_k(y) := 3.$$

Finally, let $V : \mathcal{J} \times \mathcal{B}(\Delta(C)) \longrightarrow \mathbb{R}$ be defined by

$$V(j, A) := \max_{p \in A} (p(x)u_j(x) + p(y)u_j(y))$$
$$V(k, A) := \left(\max_{p \in A} (p(x)u_k(x) + p(y)u_k(y))\right)^2$$

for any $A \in \mathcal{B}(\Delta(C))$. The preference $\succsim$ represented by V satisfies Axioms 2, 4, 6 and 7. However, we will now show that it does not satisfy Axiom 8. Let $A := \{x\} =: B'$ and $A' := \{y\} =: B$.[10] Notice that $V(j, A) = 9 = V(k, B)$, $V(j, A') = 1 = V(k, B')$, but $V(j, \frac{1}{2}A + \frac{1}{2}A') = 5 > 4 = V(k, \frac{1}{2}B + \frac{1}{2}B')$. That is, $(j, A) \sim (k, B)$, $(j, A') \sim (k, B')$, but $(j, \frac{1}{2}A + \frac{1}{2}A') \succ (k, \frac{1}{2}B + \frac{1}{2}B')$. Therefore, Axiom 8 is not satisfied, which shows that $\succsim$ does not admit a Monotone Additive EU representation. ||

We argued above that Axiom 8 is necessary for a Monotone Additive representation. The result below shows that, together with the other postulates considered in Example 2, it is also sufficient.

Theorem 3. *A preference relation $\succsim$ satisfies Axioms 2, 4, 6, 7 and 8 if, and only if, it admits a Monotone Additive EU representation.*

6 Finite Sets of Information Nodes

A large part of the motivation and interpretation of the results in Nakata (2011) is done under the assumption that the sets of information nodes $\{\mathcal{M}_j\}_{j \in \mathcal{J}}$ all have a finite number of elements. In this section we show how to guarantee that this is the case, for Ordinal and Monotone Additive EU representations. Consider the following postulate:

Axiom 9 (Finiteness)**.** *For every menu $A \in \mathcal{B}(\Delta(C))$ and every $j \in \mathcal{J}$, there exists a finite menu $B \subseteq A$ such that $(j, A) \sim (j, B)$.*

We can show the following:

[10] Here we are using the standard abuse of notation of writing x and y to represent the degenerate lotteries that assign probability one to the alternative x and y, respectively.

Proposition 1. *Suppose $\succsim$ is a preference relation that admits an Ordinal EU representation or a Monotone Additive EU representation. In this case, $\succsim$ satisfies Axiom 9 iff it admits a finite Ordinal EU representation or a finite Monotone Additive EU representation, respectively.*

Remark 1. *Dekel, Lipman, and Rustichini (2009) present an axiom that guarantees the finiteness of the state space for additive representations not necessarily monotone. (See also Kopylov (2009).) We do not know if adding that postulate to Theorem 1 above would make the sets $\{\mathcal{M}_j\}_{j\in\mathcal{J}}$ finite in that theorem.*

A Proofs

A.1 Proof of Theorem 1

The proof that the axioms are necessary for the representation is left to the reader. We will show only that the axioms are sufficient for $\succsim$ to have a weak EU representation. By Axiom 4, $\succsim$ is a continuous preference relation on the compact (therefore separable) metric space $\mathcal{J}\times\mathcal{B}(\Delta(C))$. This allows us to invoke Debreu's utility representation theorem to find a continuous function $v:\mathcal{J}\times\mathcal{B}(\Delta(C))\longrightarrow\mathbb{R}$ such that, for every pair $(i,A),(j,B)\in\mathcal{J}\times\mathcal{B}(\Delta(C))$,

$$(i,A)\succsim(j,B)\Longleftrightarrow v(i,A)\geq v(j,B).$$

Now fix $j\in\mathcal{J}$ and define the relation $\succsim_j\subseteq\mathcal{B}(\Delta(C))\times\mathcal{B}(\Delta(C))$ by $A\succsim_j B$ if and only if $(j,A)\succsim(j,B)$. Theorem 1 in DLR implies that there exits a set $\mathcal{M}_j$, a state dependent function $U_j:\Delta(C)\times\mathcal{M}_j\rightarrow\mathbb{R}$ and an agregator $\gamma_j:\mathbb{R}^{\mathcal{M}_j}\rightarrow\mathbb{R}$ such that, for every $A,B\in\mathcal{B}(\Delta(C))$,

$$A\succsim_j B\iff\gamma_j\left(\left\langle\max_{p\in A}U_j(p,m_j)\right\rangle_{m_j\in\mathcal{M}_j}\right)\geq\gamma_j\left(\left\langle\max_{p\in B}U_j(p,m_j)\right\rangle_{m_j\in\mathcal{M}_j}\right),$$

and, for every $m_j\in\mathcal{M}_j$, $U_j(.,m_j)$ is a nontrivial expected utility function. Moreover, since $\succsim_j$ agrees with $\succsim$ on $\{j\}\times\mathcal{B}(\Delta(C))$, it is clear that whenever $\gamma_j\left(\langle\max_{p\in A}U(p,m_j)\rangle_{m_j\in\mathcal{M}_j}\right)=\gamma_j\left(\langle\max_{p\in B}U(p,m_j)\rangle_{m_j\in\mathcal{M}_j}\right)$ for some $A,B\in\mathcal{B}(\Delta(C))$, we have $v(j,A)=v(j,B)$. We can also assume, without loss of generality, that for any distinct i and j in $\mathcal{J}$, $\mathcal{M}_i\cap\mathcal{M}_j=\emptyset$. This now allows us to define $\varphi_j:\mathbb{R}^{\mathcal{M}_j}\rightarrow\mathbb{R}$ by $\varphi_j(\xi):=v(j,A)$, if there exists $A\in\mathcal{B}(\Delta(C))$ with $\langle\max_{p\in A}U_j(p,m_j)\rangle_{m_j\in\mathcal{M}_j}=\xi$ and $\varphi_j(\xi):=0$ otherwise. Finally, define $U:\Delta(C)\times\cup_{j\in\mathcal{J}}\mathcal{M}_j\longrightarrow\mathbb{R}$ by $U(p,m_j):=U_j(p,m_j)$ for every $j\in\mathcal{J}$ and $m_j\in\mathcal{M}_j$. Since, for every

$j \in \mathcal{J}$ and $A \in \mathcal{B}(\Delta(C))$, $V(j,A) = \varphi_j(\langle \max_{p \in A} U(p, m_j)\rangle_{m_j \in \mathcal{M}_j})$, $(\{\mathcal{M}_j, \varphi_j\}_{j \in \mathcal{J}}, U)$ is a Weak EU representation of $\succsim$. ■

A.2 Proof of Theorem 2

Again, we leave the proof that the axioms are necessary for the representation to the reader. To show that the axioms are sufficient for the representation, as we did in the proof of Theorem 1, we can first use Debreu's utility representation theorem to find a continuous function $v : \mathcal{J} \times \mathcal{B}(\Delta(C)) \longrightarrow \mathbb{R}$ such that, for every pair $(i, A), (j, B) \in \mathcal{J} \times \mathcal{B}(\Delta(C))$,

$$(i, A) \succsim (j, B) \Longleftrightarrow v(i, A) \geq v(j, B).$$

Now fix $j \in \mathcal{J}$ and define the relation $\succsim_j \subseteq \mathcal{B}(\Delta(C)) \times \mathcal{B}(\Delta(C))$ by $A \succsim_j B$ if and only if $v(j, A) \geq v(j, B)$. Since v represents $\succsim$, $\succsim_j$ satisfies all the axioms in DLR's Theorem 3. This implies that there exits a set $\mathcal{M}_j$, a state dependent function $U_j : \Delta(C) \to \mathbb{R}$ and a strictly increasing agregator (on the relevant domain) $\gamma_j : \mathbb{R}^{\mathcal{M}_j} \to \mathbb{R}$ such that, for every $A, B \in \mathcal{B}(\Delta(C))$,

$$A \succsim_j B \iff \gamma_j\left(\left\langle \max_{p \in A} U_j(p, m_j)\right\rangle_{m_j \in \mathcal{M}_j}\right) \geq \gamma_j\left(\left\langle \max_{p \in B} U_j(p, m_j)\right\rangle_{m_j \in \mathcal{M}_j}\right),$$

and, for every $m_j \in \mathcal{M}_j$, $U_j(., m_j)$ is an expected utility function. Again, we can assume, without loss of generality, that for any distinct i and j in $\mathcal{J}$, $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$. Repeating the steps in the proof of Theorem 1, we obtain a Weak EU representation, $(\{\mathcal{M}_j, \varphi_j\}_{j \in \mathcal{J}}, U)$ of $\succsim$, with the additional restriction that all the aggregators φ_j are strictly increasing on their relevant domain. That is, $(\{\mathcal{M}_j, \varphi_j\}_{j \in \mathcal{J}}, U)$ is an Ordinal EU representation of $\succsim$. ■

A.3 Proof of Theorem 3

Once more, we will leave the proof that the axioms are necessary for the representation to the reader. We will show by induction on the number of elements of $\mathcal{J}$ that they are also sufficient. If $|\mathcal{J}| = 1$, then we are in DLR's setup and the representation comes from Theorem 2 in Dekel, Lipman, Rustichini, and Sarver (2007). Suppose now that the characterization is true when $|\mathcal{J}| = n$ and consider a setup with $|\mathcal{J}| = n + 1$. Without loss of generality, we can assume that $\mathcal{J} := \{1, ..., n + 1\}$. By Continuity and the compactness of $\mathcal{B}(\Delta(C))$, we know that for each $j \in \mathcal{J}$ there exists $\underline{A}_j \in \mathcal{B}(\Delta(C))$ such that $(j, B) \succsim (j, \underline{A}_j)$ for every $B \in \mathcal{B}(\Delta(C))$. Order the elements of J so that $\underline{A}_{n+1}$ satisfies $(n + 1, \underline{A}_{n+1}) \succsim (j, \underline{A}_j)$

for every $j \in \mathcal{J}$. Now consider the restriction of $\succsim$ to $(\mathcal{J} \setminus \{n+1\}) \times \mathcal{B}(\Delta(C))$. By the induction hypothesis, this restriction admits a Monotone Additive EU representation $(\{(\mathcal{M}_j, \Sigma_j), \mu_j\}_{j \in (\mathcal{J} \setminus \{n+1\})}, U)$. For each $j \in \mathcal{J} \setminus \{n+1\}$ and $A \in \mathcal{B}(\Delta(C))$, let

$$V(j, A) := \int_{\mathcal{M}_j} \max_{p \in A} U(p, m_j) \mu_j(dm_j).$$

Suppose first that $(n+1, \underline{A}_{n+1}) \succsim (j, \Delta(C))$ for every $j \in \mathcal{J} \setminus \{n+1\}$. If there exists $j \in \mathcal{J} \setminus \{n+1\}$ such that $(n+1, \underline{A}_{n+1}) \sim (j, \Delta(C))$, then take any Monotone Additive EU representation $((\mathcal{M}_{n+1}, \Sigma_{n+1}), \mu_{n+1}, \tilde{U})$ of the restriction of $\succsim$ to $\{n+1\} \times \mathcal{B}(\Delta(C))$ and normalize $\tilde{U}$ so that

$$\tilde{V}(n+1, \underline{A}_{n+1}) := \int_{\mathcal{M}_{n+1}} \max_{p \in \underline{A}_{n+1}} \tilde{U}(p, m_{n+1}) \mu_{n+1}(dm_{n+1}) = V(j, \Delta(C)).^{11}$$

If $(n+1, \underline{A}_{n+1}) \succ (j, \Delta(C))$ for every $j \in \mathcal{J} \setminus \{n+1\}$, then take any Monotone Additive EU representation $((\mathcal{M}_{n+1}, \Sigma_{n+1}), \mu_{n+1}, \tilde{U})$ of the restriction of $\succsim$ to $\{n+1\} \times \mathcal{B}(\Delta(C))$ and normalize $\tilde{U}$ so that

$$\tilde{V}(n+1, \underline{A}_{n+1}) := \int_{\mathcal{M}_{n+1}} \max_{p \in \underline{A}_{n+1}} \tilde{U}(p, m_{n+1}) \mu_{n+1}(dm_{n+1}) > V(j, \Delta(C))$$

for every $j \in \mathcal{J}$.[12] Now define $\hat{U} : \Delta(C) \times \cup_{j \in \mathcal{J}} \mathcal{M}_j \to \mathbb{R}$ by $\hat{U}(., j) := U(., j)$ if $j \in \mathcal{J} \setminus \{n+1\}$ and $\hat{U}(., n+1) := \tilde{U}$. It is clear that $(\{(\mathcal{M}_j, \Sigma_j), \mu_j\}_{j \in \mathcal{J}}, \hat{U})$ is a Monotone Additive EU representation of $\succsim$.

Now suppose that there exists $j \in \mathcal{J} \setminus \{n+1\}$ such that $(j, \Delta(C)) \succ (n+1, \underline{A}_{n+1})$. Let $\mathcal{I} := \{j \in \mathcal{J} \setminus \{n+1\} : (j, \Delta(C)) \succ (n+1, \underline{A}_{n+1})\}$. Since $\mathcal{J}$ is a finite set, Continuity II and Nontriviality guarantee that there exists $A^* \in \mathcal{B}(\Delta(C))$ such that $(j, \Delta(C)) \succ (n+1, A^*) \succ (n+1, \underline{A}_{n+1}) \succsim (j, \underline{A}_j)$ for every $j \in \mathcal{I}$. Now define $\mathcal{B}^* := \{A \in \mathcal{B}(\Delta(C)) : (n+1, A^*) \succsim (n+1, A)\}$. We note that the fact that the restriction of $\succsim$ to $\{n+1\} \times \mathcal{B}(\Delta(C))$ admits a Monotone Additive EU representation implies that $\mathcal{B}^*$ is a convex set. Now, for every $A \in \mathcal{B}^*$ let $W(A) := V(j, B)$ for any $j \in \mathcal{J} \setminus \{n+1\}$ and any $B \in \mathcal{B}(\Delta(C))$ such that $(n+1, A) \sim (j, B)$. A standard argument using Axiom 4 guarantees that, for any $j \in \mathcal{I}$,

[11] For example, pick a representation $((\mathcal{M}_{n+1}, \Sigma_{n+1}), \mu_{n+1}, \tilde{U})$ of the restriction of $\succsim$ to $\{n+1\} \times \mathcal{B}(\Delta(C))$ where μ_{n+1} is a probability measure and define $U' := \tilde{U} - \int_{\mathcal{M}_{n+1}} \max_{p \in \underline{A}_{n+1}} \tilde{U}(p, m_{n+1}) \mu_{n+1}(dm_{n+1}) + V(j, \Delta(C))$ and replace $\tilde{U}$ by U' in the representation.

[12] For example, pick a representation $((\mathcal{M}_{n+1}, \Sigma_{n+1}), \mu_{n+1}, \tilde{U})$ of the restriction of $\succsim$ to $\{n+1\} \times \mathcal{B}(\Delta(C))$ where μ_{n+1} is a probability measure and define $U' := \tilde{U} - \int_{\mathcal{M}_{n+1}} \max_{p \in \underline{A}_{n+1}} \tilde{U}(p, m_{n+1}) \mu_{n+1}(dm_{n+1}) + \max_{j \in \mathcal{J} \setminus \{n+1\}} V(j, \Delta(C)) + 1$ and replace $\tilde{U}$ by U' in the representation.

such a B exists. We proceed through several claims.

Claim 1. *W is affine in $\mathcal{B}^*$.*

Proof of Claim. Fix $A, B \in \mathcal{B}^*$, $\lambda \in (0,1)$ and $j \in \mathcal{I}$. By Axiom 4, there exist $A', B' \in \mathcal{B}(\Delta(C))$ such that $(n+1, A) \sim (j, A')$ and $(n+1, B) \sim (j, B')$. By definition, $W(A) = V(j, A')$ and $W(B) = V(j, B')$. By Axiom 8, we also have that $(n+1, \lambda A + (1-\lambda)B) \sim (j, \lambda A' + (1-\lambda)B')$. Since $\mathcal{B}^*$ is convex, $\lambda A + (1-\lambda)B \in \mathcal{B}^*$. Since $V(j, .)$ is affine, by the definition of W, we have $W(\lambda A + (1-\lambda)B) = V(j, \lambda A' + (1-\lambda)B') = \lambda V(j, A') + (1-\lambda)V(j, B') = \lambda W(A) + (1-\lambda)W(B)$. ‖

It turns out that W has a unique affine extension to $\mathcal{B}(\Delta(C))$.[13] Let's abuse notation and call this extension W again. We need the following three claims.

Claim 2. *W represents the restriction of $\succsim$ to $\{n+1\} \times \mathcal{B}(\Delta(C))$.*

Proof of Claim. Fix $A, B \in \mathcal{B}(\Delta(C))$ and pick λ large enough so that $\lambda \underline{A}_{n+1} + (1-\lambda)A$ and $\lambda \underline{A}_{n+1} + (1-\lambda)B$ belong to $\mathcal{B}^*$. By the definition of W and its affinity, we have that $(n+1, \lambda \underline{A}_{n+1} + (1-\lambda)A) \succsim (n+1, \lambda \underline{A}_{n+1} + (1-\lambda)B)$ iff $W(\lambda \underline{A}_{n+1} + (1-\lambda)A) \geq W(\lambda \underline{A}_{n+1} + (1-\lambda)B)$ iff $W(A) \geq W(B)$. Now, since the restriction of $\succsim$ to $\{n+1\} \times \mathcal{B}(\Delta(C))$ admits a Finite Monotone Additive EU representation, we also have that $(n+1, A) \succsim (n+1, B)$ iff $(n+1, \lambda \underline{A}_{n+1} + (1-\lambda)A) \succsim (n+1, \lambda \underline{A}_{n+1} + (1-\lambda)B)$, which completes the proof of the claim. ‖

Claim 3. *Fix $A \in \mathcal{B}(\Delta(C))$. If there exists $j \in \mathcal{J} \setminus \{n+1\}$ and $B \in \mathcal{B}(\Delta(C))$ such that $(n+1, A) \sim (j, B)$, then $W(A) = V(j, B)$. If $(n+1, A) \succ (j, B)$ for every $j \in \mathcal{J} \setminus \{n+1\}$ and $B \in \mathcal{B}(\Delta(C))$, then $W(A) > V(j, B)$ for every $j \in \mathcal{J} \setminus \{n+1\}$ and $B \in \mathcal{B}(\Delta(C))$.*

Proof of Claim. The claim is immediate if $A \in \mathcal{B}^*$. Suppose, thus, that $A \notin \mathcal{B}^*$ and $(n+1, A) \sim (j, B)$ for some $j \in \mathcal{J} \setminus \{n+1\}$ and some $B \in \mathcal{B}(\Delta(C))$. Axiom 4 implies that there exists $\underline{B} \in \mathcal{B}(\Delta(C))$ such that $(n+1, \underline{A}_{n+1}) \sim (j, \underline{B})$. Now pick $\lambda \in (0,1)$ large enough so that $\lambda \underline{A}_{n+1} + (1-\lambda)A \in \mathcal{B}^*$. By Axiom 8, we have $(n+1, \lambda \underline{A}_{n+1} + (1-\lambda)A) \sim (j, \lambda \underline{B} + (1-\lambda)B)$, which implies that $W(\lambda \underline{A}_{n+1} + (1-\lambda)A) = V(j, \lambda \underline{B} + (1-\lambda)B)$. Since both W and $V(j, .)$ are affine, and $W(\underline{A}_{n+1}) = V(\underline{B})$, we get that $W(A) = V(j, B)$. Finally, suppose that $(n+1, A) \succ (j, B)$ for every $j \in \mathcal{J} \setminus \{n+1\}$ and $B \in \mathcal{B}(\Delta(C))$.

[13] To show the uniqueness part of this claim,suppose that $\hat{W}$ is an affine extension of W and fix $A \in \mathcal{B}(\Delta(C))$. Pick $\lambda \in (0,1)$ large enough so that $\lambda \underline{A}_{n+1} + (1-\lambda)A \in \mathcal{B}^*$. By construction, $\lambda \hat{W}(\underline{A}_{n+1}) + (1-\lambda)\hat{W}(A) = \hat{W}(\lambda \underline{A}_{n+1} + (1-\lambda)A) = W(\lambda \underline{A}_{n+1} + (1-\lambda)A)$. Since $\hat{W}(\underline{A}_{n+1}) = W(\underline{A}_{n+1})$, this implies that $\hat{W}(A) = \frac{W(\lambda \underline{A}_{n+1} + (1-\lambda)A) - \lambda W(\underline{A}_{n+1})}{1-\lambda}$, which shows that $\hat{W}$ is unique.

Fix $j^* \in \mathcal{J} \setminus \{n+1\}$ and $B^* \in \mathcal{B}(\Delta(C))$. If there exists $A^* \in \mathcal{B}(\Delta(C))$ such that $(n+1, A^*) \sim (j, B^*)$, then, by the previous claim and the previous observation, we have $W(A) > W(A^*) = V(j, B^*)$. Otherwise, we have $W(A) > W(\underline{A}_{n+1}) > V(j, B^*)$. ||

Claim 4. *There exists a Monotone Additive EU representation $((\mathcal{M}_{n+1}, \Sigma_{n+1}), \mu_{n+1}, \tilde{U})$ of the restriction of $\succsim$ to $\{n+1\} \times \mathcal{B}(\Delta(C))$ such that*

$$W(A) := \int_{\mathcal{M}_{n+1}} \max_{p \in A} \tilde{U}(p, m_{n+1}) \mu_{n+1}(dm_{n+1}),$$

for every $A \in \mathcal{B}(\Delta(C))$.

Proof of Claim. By theorem 2 in Dekel et al. (2007), we know that the restriction of $\succsim$ to $\{n+1\} \times \mathcal{B}(\Delta(C))$ admits a Monotone Additive EU representation $((\mathcal{M}_{n+1}, \Sigma_{n+1}), \mu_{n+1}, \hat{U})$. Moreover, we can choose this representation so that μ_{n+1} is a probability measure. Let $\hat{V} : \mathcal{B}(\Delta(C)) \to \mathbb{R}$ be defined by

$$\hat{V}(A) := \int_{\mathcal{M}_{n+1}} \max_{p \in A} \hat{U}(p, m_{n+1}) \mu_{n+1}(dm_{n+1}), \text{ for every } A \in \mathcal{B}(\Delta(C)).$$

Now notice that both W and $\hat{V}$ are affine representations of the restriction of $\succsim$ to $\{n+1\} \times \mathcal{B}(\Delta(C))$. The uniqueness properties of such representations imply that there exists $\alpha \in \mathbb{R}_{++}$ and $\beta \in \mathbb{R}$ such that $W(A) = \alpha \hat{V}(A) + \beta$ for every $A \in \mathcal{B}(\Delta(C))$. Define $\tilde{U} : \Delta(C) \to \mathbb{R}$ by $\tilde{U}(p) := \alpha \hat{U}(p) + \beta$ for every $p \in \Delta(C)$. Notice that

$$W(A) = \int_{\mathcal{M}_{n+1}} \max_{p \in A} \tilde{U}(p, m_{n+1}) \mu_{n+1}(dm_{n+1}), \text{ for every } A \in \mathcal{B}(\Delta(C)).$$

This concludes the proof of the claim. ||

Now define $\tilde{V} : \mathcal{J} \times \mathcal{B}(\Delta(C)) \to \mathbb{R}$ by $\tilde{V}(j, .) := V(j, .)$ if $j \in \mathcal{J} \setminus \{n+1\}$ and $\tilde{V}(n+1, .) := W$. The previous three claims imply that $\tilde{V}$ represents $\succsim$ and that it admits a Monotone Additive EU representation. ■

A.4 Proof of Proposition 1

Suppose first that $\succsim$ admits an Ordinal EU representation $(\{\mathcal{M}_j, \varphi_j\}_{j \in \mathcal{J}}, U)$. We can assume, without loss of generality, that the sets $\{\mathcal{M}_j\}_{j \in \mathcal{J}}$ have no redundant nodes. That is, for every $j \in \mathcal{J}$ and every distinct m_j and m'_j in $\mathcal{M}_j$, $U(., m_j)$ and $U(., m'_j)$ represent different nontrivial expected utility preferences. We now show that Finiteness implies that all the sets $\mathcal{M}_j$ are finite. Fix $j \in \mathcal{J}$ and pick any sphere $A \in \mathcal{B}(\Delta(C))$. That is, pick $A \in \mathcal{B}(\Delta(C))$ such that there exists $p \in \Delta(C)$ and $\delta > 0$ such that $A := \{q \in \mathbb{R}^C : \|q - p\| \leq \delta \text{ and } \sum_{c \in C} q(c) = 1\}$.[14] Because A is a sphere, for every pair m_j and m'_j in $\mathcal{M}_j$ we have that $\arg\max_{p \in A} U(p, m_j) \neq \arg\max_{p \in A} U(p, m'_j)$ and both $\arg\max_{p \in A} U(p, m_j)$ and $\arg\max_{p \in A} U(p, m'_j)$ are singletons. Now suppose that $\mathcal{M}_j$ is not a finite set for some $j \in \mathcal{J}$ and pick any finite $B \in \mathcal{B}(\Delta(C))$ such that $B \subseteq A$. By our previous observation, there exists $m_j \in \mathcal{M}_j$ such that $\max_{p \in B} U(p, m_j) < \max_{p \in A} U(p, m_j)$. Since φ_j is strictly increasing and $(\{\mathcal{M}_j, \varphi_j\}_{j \in \mathcal{J}}, U)$ represents $\succsim$, this implies that $(j, A) \succ (j, B)$. That is, for no finite subset B of A we can have $(j, A) \sim (j, B)$, which contradicts Finiteness. We conclude that $\mathcal{M}_j$ must be a finite set.

Now suppose that the preference relation $\succsim$ admits a Monotone Additive EU representation $(\{(\mathcal{M}_j, \Sigma_j), \mu_j\}_{j \in \mathcal{J}}, U)$. From Dekel et al. (2007), we know that $(\{(\mathcal{M}_j, \Sigma_j), \mu_j\}_{j \in \mathcal{J}}, U)$ can be chosen so that, for each $j \in \mathcal{J}$, $\mathcal{M}_j$ is a metric space, Σ_j is the collection of Borel subsets of $\mathcal{M}_j$, μ_j is a probability measure on $(\mathcal{M}_j, \Sigma_j)$ whose support is exactly $\mathcal{M}_j$ and, for each menu $A \in \mathcal{B}(\Delta(C))$, the function $\sigma^j_A : \mathcal{M}_j \to \mathbb{R}$ defined by $\sigma^j_A(m_j) := \max_{p \in A} U(m_j)$ for every $m_j \in \mathcal{M}_j$, is continuous. Now, for each $j \in \mathcal{J}$, define $\varphi_j : \mathbb{R}^{\mathcal{M}_j} \to \mathbb{R}$ by

$$\varphi_j(\xi) := \int_{\mathcal{M}_j} \xi(m_j) \mu_j(dm_j),$$

if there exists $A \in \mathcal{B}(\Delta(C))$ with $\langle \max_{p \in A} U_j(p, m_j) \rangle_{m_j \in \mathcal{M}_j} = \xi$ and $\varphi_j(\xi) := 0$ otherwise. Given our choice of Monotone Additive EU representation for $\succsim$, it turns out that $(\{\mathcal{M}_j, \varphi_j\}_{j \in \mathcal{J}}, U)$ is an Ordinal EU representation of $\succsim$. Now the first part of this proof implies that all the sets in $\{\mathcal{M}_j\}_{j \in \mathcal{J}}$ must be finite. ■

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[14] By $\|q - p\|$ we mean $\sqrt{\sum_{x \in C} (q(x) - p(x))^2}$.

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