TOPOLOGICAL CLOSURE OF TRANSLATION INVARIANT PREORDERS

Efe A. Ok and Gil Riella

Abstract. Our primary query is to find conditions under which the closure of a preorder on a topological space remains transitive. We study this problem for translation invariant preorders on topological groups. The results are fairly positive; we find that the closure of preorders and normal-orders remain as such in this context. The same is true for factor orders as well under quite general conditions. In turn, in the context of topological linear spaces, these results allow us to obtain a simple condition under which the order-duals with respect to a vector order and its closure coincide. Various order-theoretic applications of these results are also provided in the paper.

1. INTRODUCTION

Continuous preorders on a topological space X – by continuity here we mean being closed in $X \times X$ – play an essential role in applications of order theory to optimization theory, topology and functional analysis. For instance, the existence of extremal elements with respect to a preorder on X is often established by appealing to the continuity of that preorder. Similarly, Nachbin-Urysohn type monotonic extension theorems (cf. [**15**], [**12**], and Chapter 5 of [**3**]), and Carruth-Urysohn type metrization theorems (cf. [**5**]), are based on this continuity property. In addition, continuity of preorders plays a fundamental role in economics. In particular, many multi-utility representation theorems for preorders (interpreted as "incomplete preference relations") are obtained on the basis of continuity of those preorders. (See, for instance, [**6**], [**7**], [**16**], and [**20**].)

When one is given a preorder $\succsim$ on X which need not be continuous, the next best thing is often to minimally extend $\succsim$ to a continuous preorder on X. Taking the topological closure of $\succsim$ in $X \times X$ is a natural strategy in this regard, but, unfortunately, transitivity of $\succsim$ is in general not preserved under this operation. This is the case even when $\succsim$ is antisymmetric. For instance, let $\succsim$ be the partial order on $\mathbb{R}$ whose asymmetric part $\succ$ is given as follows: $s \succ t$ iff $(s,t) = (1,2)$ or $(s,t) = (3,0)$ or $1 > t > s > 0$ or $3 > t > s > 2$. Then, the closure of $\succsim$ in $\mathbb{R}^2$ – let us denote this by $\trianglerighteq$ – is an antisymmetric binary relation on $\mathbb{R}$ such that $1 \trianglerighteq$

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$2 \trianglerighteq 3 \trianglerighteq 0 \trianglerighteq 1$. Thus, the closure of a partial order (on $\mathbb{R}$) need not be acyclic, let alone transitive.

Positive results concerning the transitivity of the "closure of a preorder" can apparently be obtained only in the presence of further structure. In this paper we use an additional algebraic structure to this end. In particular, we focus on translation invariant preorders on topological groups and vector preorders on topological linear spaces. This property is naturally motivated from an algebraic viewpoint. Indeed, in the theory of partially ordered groups, it is precisely this property that brings together the group and order structures that are imposed on a given set. From a more applied point of view, we note that translation invariance is widely used in welfare economics to derive certain types of (utilitarian) social welfare functions/relations on the space of (finite or infinite) cardinal utility vectors (see [**14**] and references cited therein).

Our main findings can be summarized as follows. After going through some preliminaries, we show in Section 3.1 that the closure of a translation invariant preorder on a topological group is transitive, that is, a preordered topological group remains as such relative to the closure of its preorder. The same does not hold for partially ordered topological groups. This is because the closure of a partial order $\succcurlyeq$ need not be a partial order in this context, for it need not inherit the antisymmetry of $\succcurlyeq$. We report two general conditions under which this difficulty would not arise, however. First, we show in Section 3.2 that if the original partial order $\succcurlyeq$ is normal (that is, there is an order-convex neighborhood base at the identity of the group), then the closure of $\succcurlyeq$ is also a partial order (as well as normal). Second, we demonstrate in Section 3.3 that the preordered factor group induced by a preordered Hausdorff topological group and a compact and normal subgroup is bound to be partially ordered (even though the preorder of the ambient group is not antisymmetric), provided that the subgroup at hand is order-convex with respect to the closure of $\succcurlyeq$.

As simple as they are, the results of Section 3 seem to have a scope for applications; in Section 4 we provide some applications to functional analysis, topological order theory, utility theory, optimization, and combinatorics. In particular, in Section 4.1, we show that, under the hypotheses of local convexity and solidity, the order-dual of a normally ordered topological linear space remains intact relative to the closure of that order. In Section 4.2, we provide some continuous extension theorems for preorders on topological groups. In Section 4.3 we derive a utility representation theorem for incomplete preference relations, and in Section 4.4, we obtain sufficient conditions for the existence of extrema in a set with respect to a given translation invariant partial order on a topological group. Finally, in Section 4.5, we prove that at most 4 relations can be generated from a given translation invariant binary relation by taking topological and transitive closures. It is also shown that this is the best bound.

2. PRELIMINARIES

Let X be a nonempty set. By a *binary relation* on X, we mean any nonempty subset of $X \times X$. For any binary relation R on X, we adopt the usual convention of writing $x \; R \; y$ instead of $(x, y) \in R$. The *inverse* of R is itself a binary relation on X defined as $R^{-1} := \{(y, x) : x \; R \; y\}$. Moreover, for any binary relations R and S on X, we simply write $x \; R \; y \; S \; z$ to mean $x \; R \; y$ and $y \; S \; z$, and so on. For any subset A

of X, the *increasing closure* of A with respect to R is defined as $A^{\uparrow,R} := \{x \in X : x\ R\ y$ for some $y \in A\}$. The *decreasing closure* of A with respect to R is then defined as $A^{\uparrow,R^{-1}}$; we denote this set by $A^{\downarrow,R}$. (For any $x \in X$, we write $x^{\uparrow,R}$ for $\{x\}^{\uparrow,R}$ and $x^{\downarrow,R}$ for $\{x\}^{\downarrow,R}$.) In addition, we say that A is *R-convex* if for any $x, y \in A$ with $x\ R\ z\ R\ y$ we have $z \in A$. That is, A is R-convex iff $A \supseteq A^{\uparrow,R} \cap A^{\downarrow,R}$. Finally, A is said to be *R-bounded* if $A \subseteq x^{\downarrow,R} \cap y^{\uparrow,R}$ for some $x, y \in X$, while a set of the form $x^{\downarrow,R} \cap y^{\uparrow,R}$ is called an *R-interval*.

The *asymmetric part* of a binary relation R on X is defined as $P_R := R \backslash R^{-1}$ and the *symmetric part* of R is $I_R := R \cap R^{-1}$. The *composition* of two binary relations R and S on X is defined as $R \circ S := \{(x, y) \in X \times X : x\ R\ z\ S\ y$ for some $z \in X\}$. In turn, we let $R^1 := R$ and $R^n := R \circ R^{n-1}$ for any integer $n > 1$; here R^n is said to be the *nth iterate* of R.

We denote the diagonal of $X \times X$ by $\triangle_X$. A binary relation R on X is said to be *reflexive* if $\triangle_X \subseteq R$, *antisymmetric* if $R \cap R^{-1} \subseteq \triangle_X$, *transitive* if $R \circ R \subseteq R$, and *total* (or *complete*, or *linear*) if $R \cup R^{-1} = X \times X$. If R is reflexive and transitive, we refer to it as a *preorder* on X. If it is an antisymmetric preorder, we call it a *partial order* on X, and if it is a total partial order, we call it a *linear order.* The pair (X, R) is called a *preordered set* if R is a preorder on X, a *poset* if R is a partial order on X, and a *loset* if R is a linear order on X. (Throughout the paper, a generic preorder is denoted as $\succsim$, and a generic partial order as $\succcurlyeq$.) Finally, we say that R is *acyclic* if $\triangle_X \cap P_R^n = \emptyset$ for every positive integer n. It is readily verified that transitivity of a binary relation implies its acyclicity, but not conversely.

When X is a topological space, we say that a binary relation R on X is *continuous* if R is a closed subset of $X \times X$, and that it is *open-continuous* if the asymmetric part of R is an open subset of $X \times X$ (relative to the product topology). The closure of R in $X \times X$ is denoted by $\mathrm{cl}(R)$. In turn, where $\succsim$ is a preorder (partial order), we refer to $(X, \succsim)$ as a *preordered* (*partially ordered*) *topological space* if X is a topological space. If, in addition, $\succsim$ is continuous, we say that $(X, \succsim)$ is a *continuously preordered topological space*, but, as usual, we refer to a continuously partially ordered topological space as a *pospace.* It is well known that if $(X, \succsim)$ is a pospace, then X must be Hausdorff. (See, for instance, Proposition 2 of [**15**].) This is false in the case of continuously preordered topological spaces. (Indeed, $(X, X \times X)$ is such a space for any topological space X.)

Let X be a group, which we notate multiplicatively, and R a reflexive binary relation on X. (As usual, $\mathbf{1}$ stands for the neutral element of this group, and for any $x \in X$ and $A, B \subseteq X$, we set $AB := \{ab : (a, b) \in A \times B\}$, $Ax := A\{x\}$, $xB := \{x\}B$, and $A^{-1} := \{a^{-1} : a \in A\}$.) We define

$$C(R) := \{x \in X : x\ R\ \mathbf{1}\},$$

and recall that R is called *translation invariant* on X if

$$x\ R\ y \quad \text{implies} \quad x\omega\ R\ y\omega \text{ and } \omega x\ R\ \omega y$$

for every $x, y, \omega \in X$. When R is translation invariant, we have

$$A^{\uparrow,R} = AC(R) = C(R)A \quad \text{and} \quad A^{\downarrow,R} = AC(R)^{-1} = C(R)^{-1}A \tag{2.1}$$

for any subset A of X. (For instance, if $x \in A^{\uparrow,R}$, then $x\ R\ y$ for some $y \in A$, so by translation invariance of R, we have $y^{-1}x \in C(R)$, and hence $x = y(y^{-1}x) \in AC(R)$. Conversely, if $x = yz$ for some $(y, z) \in A \times C(R)$, then translation invariance of R and $z\ R\ \mathbf{1}$ imply $x = yz\ R\ y\mathbf{1} = y$, that is, $x \in A^{\uparrow,R}$. Thus: $A^{\uparrow,R} = AC(R)$.)

Note also that $C(R)$ is a normal submonoid of X, that is, $\mathbf{1} \in CC \subseteq C$ and $x^{-1}Cx \subseteq C$ for every $x \in X$, provided that R is a translation invariant preorder.

We say that a preordered set $(X, \succsim)$ is a *preordered group* if X is a group and $\succsim$ is translation invariant on X. (We will often consider X as a topological group, but in that case no relation between the topology of X and $\succsim$ will be postulated; these will be connected at the outset only indirectly through the continuity of the operations of X and the translation invariance of $\succsim$.) *Po-groups* and *lattice-ordered groups* are defined similarly. For future reference, we recall that the lattice and group operations of a lattice-ordered group $(X, \succsim)$ are brought together by the fact that $(x \vee y)z = xz \vee yz$ and $(x \wedge y)z = xz \wedge yz$ for every $x, y, z \in X$.

3. CLOSURES OF PREORDERS

3.1. Closure of a Translation-Invariant Preorder. The following result collects some elementary observations about translation invariant relations on a topological group X (for which the collection of all open neighborhoods of $\mathbf{1}$ is denoted as $\mathcal{N}_X$). In what follows, $\mathrm{cl}(\cdot)$ stands for the topological closure operator. When applied to sets in $X \times X$, the presumed topology on the $X \times X$ is the product topology.

Lemma 3.1.1. *Let R be a reflexive translation invariant relation on a topological group X. Then*:

(a) $\mathrm{cl}(R)$ *is translation invariant on* X;
(b) R *is continuous iff* $C(R)$ *is closed in* X;
(c) $\mathrm{cl}(C(R)) = C(\mathrm{cl}(R))$;
(d) $C(\mathrm{cl}(R)) = \bigcap\{NC(R) : N \in \mathcal{N}_X\} = \bigcap\{C(R)N : N \in \mathcal{N}_X\}$;
(e) $x^{\uparrow,\mathrm{cl}(R)} = \mathrm{cl}(x^{\uparrow,R})$ and $x^{\downarrow,\mathrm{cl}(R)} = \mathrm{cl}(x^{\downarrow,R})$ for any $x \in X$.

Proof. It is easily checked that parts (a) and (b) follow from the continuity of the product operation $(x, y) \mapsto xy$ (from $X \times X$ onto X) and that of inversion $x \mapsto x^{-1}$ (from X onto X). In turn, (a) and (b) together entail that $C(\mathrm{cl}(R))$ is a closed subset of X. As this set contains $C(R)$, then, $\mathrm{cl}(C(R)) \subseteq C(\mathrm{cl}(R))$. On the other hand, for any given $x \in C(\mathrm{cl}(R))$, we have $x \ \mathrm{cl}(R) \ \mathbf{1}$, so there are nets (x_α) and (y_α) in X such that $x_\alpha \to x$, $y_\alpha \to \mathbf{1}$ and $x_\alpha \ R \ y_\alpha$ for each α. Then, by translation invariance, $x_\alpha y_\alpha^{-1} \in C(R)$ for each α, so by continuity of the product operation and inversion, we find $x \in \mathrm{cl}(C(R))$. It thus follows that $C(\mathrm{cl}(R)) \subseteq \mathrm{cl}(C(R))$, and part (c) is proved. In turn, part (d) follows from part (c) and the fact that $\mathrm{cl}(A) = \bigcap_{N \in \mathcal{N}_X} NA = \bigcap_{N \in \mathcal{N}_X} AN$ for any subset A of X (which is valid in any topological group).[1] Finally, for any given x in X, part (c) and (2.1) imply

$$x^{\uparrow,\mathrm{cl}(R)} = xC(\mathrm{cl}(R)) = x\mathrm{cl}(C(R)) = \mathrm{cl}(xC(R)) = \mathrm{cl}(x^{\uparrow,R})$$

and similarly for $x^{\downarrow,\mathrm{cl}(R)}$.

We now use this lemma to show that the closure of a preorder on a topological group is indeed transitive, provided that this preorder is translation invariant.

Proposition 3.1.2. *Let $(X, \succsim)$ be a preordered topological group. Then* $\mathrm{cl}(\succsim)$ *is a preorder on* X.

[1] See, for instance, [**19**], Lemma 14.11.

Proof. Take any $x, y, z \in X$ with $x \operatorname{cl}(\succsim)\, y \operatorname{cl}(\succsim)\, z$. Then, by translation invariance of $\operatorname{cl}(\succsim)$, both xy^{-1} and yz^{-1} belong to $C(\operatorname{cl}(\succsim))$, so by part (c) of Lemma 3.1.1, $\{xy^{-1}, yz^{-1}\} \subseteq \operatorname{cl}(C(\succsim))$. It follows that there exist two nets (a_α) and (b_β) in $C(\succsim)$ such that $a_\alpha \to xy^{-1}$ and $b_\beta \to yz^{-1}$. Then, by translation invariance of $\succsim$, $a_\alpha b_\beta \succsim \mathbf{1} b_\beta = b_\beta \succsim \mathbf{1}$, so by transitivity of $\succsim$, we find $a_\alpha b_\beta \in C(\succsim)$ for every α and β. As continuity of the product operation ensures $a_\alpha b_\beta \to xy^{-1}yz^{-1} = xz^{-1}$, therefore, $xz^{-1} \in \operatorname{cl}(C(\succsim))$. Then, by part (c) of Lemma 3.1.1, $xz^{-1} \operatorname{cl}(\succsim)\, \mathbf{1}$, and hence, $x \operatorname{cl}(\succsim)\, z$, as we sought.

Combining part (a) of Lemma 3.1.1 and Proposition 3.1.2 yields:

Corollary 3.1.3. *If $(X, \succsim)$ is a preordered topological group, so is $(X, \operatorname{cl}(\succsim))$.*

3.2. Closure of a Translation-Invariant Normal Order. In general, we cannot replace "preordered" with "partially ordered" in the statement of Corollary 3.1.3, because antisymmetry of $\succsim$ need not be inherited by $\operatorname{cl}(\succsim)$. For instance, $\mathbb{R}^2$ is an ordered topological group under the lexicographic order $\succcurlyeq_{\text{lex}}$ but $\operatorname{cl}(\succcurlyeq_{\text{lex}})$ is not antisymmetric, for $(x_1, x_2) \operatorname{cl}(\succcurlyeq_{\text{lex}})\, (y_1, y_2)$ iff $x_1 \geq y_1$.[2]

However, this sort of a problem does not arise under reasonable conditions. In what follows, we will show that normality of the partial order would provide an escape route in this respect. We shall later show that the difficulty is also largely alleviated in the case of preordered topological factor groups.

We say that a preorder $\succsim$ on a topological group X is *normal* if, for any two nets (x_α) and (y_α) in X with $x_\alpha \to \mathbf{1}$ and $x_\alpha \succsim y_\alpha \succsim \mathbf{1}$ for each α, we have $y_\alpha \to \mathbf{1}$. If it is translation invariant and X is Hausdorff, such a preorder must be antisymmetric, for then $x \succsim y \succsim x$ implies $\mathbf{1} \succsim x^{-1}y \succsim \mathbf{1}$, and hence, by normality of $\succsim$, $(x^{-1}y)$ is a constant net that converges to $\mathbf{1}$, which, as X is Hausdorff, means $x = y$. Consequently, we say that a partially ordered Hausdorff topological group $(X, \succcurlyeq)$ is *normally ordered* if $\succcurlyeq$ is normal. (It is well known that $(X, \succcurlyeq)$ is normally ordered iff there is a base of $\succsim$-convex neighborhoods of $\mathbf{1}$.[3]) We will now show that the closure of $\succcurlyeq$ remains antisymmetric in the context of any such po-group.[4]

Proposition 3.2.1. *If $(X, \succcurlyeq)$ is a normally ordered Hausdorff topological group, so is $(X, \operatorname{cl}(\succcurlyeq))$.*

We will use the following fact to prove this result.

Lemma 3.2.2. *Let $(X, \succcurlyeq)$ be a normally ordered Hausdorff topological group. Then, for every open neighborhood O of $\mathbf{1}$ there is an open neighborhood U of $\mathbf{1}$ such that $U^{\uparrow, \operatorname{cl}(\succcurlyeq)} \cap U^{\downarrow, \operatorname{cl}(\succcurlyeq)} \subseteq O$.*

Proof. We will first establish the desired property for $\succcurlyeq$, and then extend it to the case of $\operatorname{cl}(\succcurlyeq)$. Let $\mathcal{N}_X$ be the collection of all open neighborhoods of $\mathbf{1}$. To derive a contradiction, suppose we could find an open neighborhood O of $\mathbf{1}$ such that there

[2] By definition, we have $(x_1, x_2) \succcurlyeq_{\text{lex}} (y_1, y_2)$ iff either $x_1 > y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$.

[3] See, for example, [**1**], Theorem 2.23.

[4] One may wonder if this observation really has some bite. That is, one may wonder if a translation invariant normal preorder is not continuous to start with. In general, these two properties are not related. For example, the relation $\succcurlyeq \subseteq \mathbb{R} \times \mathbb{R}$ defined by $x \succcurlyeq y$ iff $x = y$ or $x - y > 1$ is a translation invariant normal preorder that is not continuous. In fact, its closure is the normal preorder such that $x \operatorname{cl}(\succcurlyeq) y$ iff $x = y$ or $x - y \geq 1$.

is some a_V in $(V^{\uparrow,\succcurlyeq} \cap V^{\downarrow,\succcurlyeq})\backslash O$ for every $V \in \mathcal{N}_X$. Then, for any $V \in \mathcal{N}_X$, there is an $(x_V, y_V) \in V \times V$ such that $x_V \succcurlyeq a_V \succcurlyeq y_V$ while $a_V \in X\backslash O$. Furthermore, as $\mathcal{N}_X$ is a directed set under the reverse containment ordering, we can regard (x_V), (a_V) and (y_V) as nets in X where V varies over $\mathcal{N}_X$. Clearly, we have $(x_V, y_V) \to (\mathbf{1}, \mathbf{1})$, so $x_V y_V^{-1} \to \mathbf{1}$ by continuity of multiplication and inversion. Furthermore, $x_V y_V^{-1} \succcurlyeq a_V y_V^{-1} \succcurlyeq \mathbf{1}$ for each $V \in \mathcal{N}_X$, so normality ensures that $a_V y_V^{-1} \to \mathbf{1}$. Then $a_V = (a_V y_V^{-1}) y_V \to \mathbf{1}$, so $a_V \in O$ for some $V \in \mathcal{N}_X$, a contradiction. We proved: For every $O \in \mathcal{N}_X$ there is a $V \in \mathcal{N}_X$ such that $V^{\uparrow,\succcurlyeq} \cap V^{\downarrow,\succcurlyeq} \subseteq O$.

Now, fix an arbitrary $O \in \mathcal{N}_X$, and use what we have just shown to find a $V \in \mathcal{N}_X$ with $V^{\uparrow,\succcurlyeq} \cap V^{\downarrow,\succcurlyeq} \subseteq O$. In turn, using the continuity of the operations of multiplication and inversion we can find an $U \in \mathcal{N}_X$ such that $U = U^{-1}$ and $UU \subseteq V$.[5] Then, by (2.1) and part (d) of Lemma 3.1.1,

$$U^{\uparrow,\mathrm{cl}(\succcurlyeq)} = UC(\mathrm{cl}(\succcurlyeq)) \subseteq U(UC(\succcurlyeq)) \subseteq VC(\succcurlyeq) = V^{\uparrow,\succcurlyeq}.$$

As a similar argument shows that $U^{\downarrow,\mathrm{cl}(\succcurlyeq)} \subseteq V^{\downarrow,\succcurlyeq}$, we find that $U^{\uparrow,\mathrm{cl}(\succcurlyeq)} \cap U^{\downarrow,\mathrm{cl}(\succcurlyeq)}$ lies within $V^{\uparrow,\succcurlyeq} \cap V^{\downarrow,\succcurlyeq}$ and hence within O, as we sought.

Proof of Proposition 3.2.1. Let $(X, \succcurlyeq)$ be a normally ordered Hausdorff topological group. By Corollary 3.1.3, $\mathrm{cl}(\succcurlyeq)$ is a translation invariant preorder on X. We are to show that this preorder is normal. To this end, take any two nets (x_α) and (y_α) in X with $x_\alpha \to \mathbf{1}$ and $x_\alpha \ \mathrm{cl}(\succcurlyeq)\ y_\alpha \ \mathrm{cl}(\succcurlyeq)\ \mathbf{1}$ for each α. (We denote here the directed (index) set of these nets by $\mathbb{A}$, and the partial order that "directs" $\mathbb{A}$ by $\trianglerighteq$.) Pick any $O \in \mathcal{N}_X$. By Lemma 3.2.2, $U^{\uparrow,\mathrm{cl}(\succcurlyeq)} \cap U^{\downarrow,\mathrm{cl}(\succcurlyeq)} \subseteq O$ for some $U \in \mathcal{N}_X$. Since $x_\alpha \to \mathbf{1}$, there is an index $\alpha_U \in \mathbb{A}$ such that $x_\alpha \in U$ for all $\alpha \in \mathbb{A}$ with $\alpha \trianglerighteq \alpha_U$. It follows that $y_\alpha \in U^{\uparrow,\mathrm{cl}(\succcurlyeq)} \cap U^{\downarrow,\mathrm{cl}(\succcurlyeq)}$, and hence $y_\alpha \in O$, for all $\alpha \in \mathbb{A}$ with $\alpha \trianglerighteq \alpha_U$. This proves that $y_\alpha \to \mathbf{1}$, and completes the proof.

As an application of Proposition 3.2.1, we provide a sufficient condition for the closure of a normal lattice-order on a topological group to be total.

Proposition 3.2.3. *Let $(X, \succcurlyeq)$ be a normally lattice-ordered Hausdorff topological group. Then, $\mathrm{cl}(\succcurlyeq)$ is a linear order on X, provided that every x in X that is isolated from $C(\succcurlyeq)$ is comparable to x^{-1} with respect to $\succcurlyeq$.*

Proof. By part (a) of Lemma 3.1.1, $\mathrm{cl}(\succcurlyeq)$ is translation invariant, so this relation is total iff $C(\mathrm{cl}(\succcurlyeq)) \cup C(\mathrm{cl}(\succcurlyeq))^{-1} = X$. Suppose $\mathrm{cl}(\succcurlyeq)$ is not total. By, part (c) of Lemma 3.1.1, then, $\mathrm{cl}(C(\succcurlyeq)) \cup (\mathrm{cl}(C(\succcurlyeq)))^{-1}$ is a proper subset of X. It follows that there is an x in $X\backslash C(\succcurlyeq)^{-1}$ which is isolated from $C(\succcurlyeq)$. We wish to show that x and x^{-1} are not comparable with respect to $\succcurlyeq$. To derive a contradiction, suppose $x \succcurlyeq x^{-1}$, which implies that $x^2 \succcurlyeq \mathbf{1}$. But then $(x \wedge \mathbf{1})^2 = x^2 \wedge x \wedge \mathbf{1} = x \wedge \mathbf{1}$, and hence $x \wedge \mathbf{1} = \mathbf{1}$, that is, $x \succcurlyeq \mathbf{1}$, a contradiction. If, on the other hand, $x^{-1} \succcurlyeq x$ were the case, a similar reasoning would yield $\mathbf{1} \succcurlyeq x$, that is, $x \in C(\succcurlyeq)^{-1}$, which is again a contradiction. Conclusion: $\mathrm{cl}(\succcurlyeq)$ is total. Combining this fact with Proposition 3.2.1 completes the proof.

[5]See, for instance, [**19**], Lemma 14.9.

3.3. Closure of a Translation-Invariant Factor Order. Let $(X, \succsim)$ be a preordered group and Z a normal subgroup of X. We define the binary relation $\succsim_Z$ on the factor group X/Z by

$$xZ \succsim_Z yZ \quad \text{iff} \quad x \succsim yz \text{ for some } z \in Z.$$

It is readily checked that $\succsim_Z$ is well-defined, and it is a preorder on X/Z; we refer to $\succsim_Z$ as the **factor preorder** induced by Z. It is worth noting that

$$x \succsim y \quad \text{implies} \quad xZ \succsim_Z yZ,$$

but not conversely, for any x and y in X. The following result recalls a few other properties of this preorder.

Lemma 3.3.1. *Let $(X, \succsim)$ be a preordered group and Z a normal subgroup of X. Then $(X/Z, \succsim_Z)$ is a preordered group. Furthermore, $\succsim_Z$ is antisymmetric iff Z is $\succsim$-convex.*

Proof. Take any x, y and ω in X such that $xZ \succsim_Z yZ$. Then $x \succsim yz$ for some $z \in Z$ so, by translation invariance of $\succsim$, we have $\omega x \succsim \omega yz$. It follows that $\omega Z x Z = (\omega x)Z \succsim_Z (\omega y)Z = \omega Z y Z$. Similarly, $x\omega \succsim yz\omega$, so, because $(yz)Z = y(zZ) = yZ$, we find $xZ\omega Z = (x\omega)Z \succsim_Z (yz\omega)Z = (yz)Z\omega Z = yZ\omega Z$. We conclude that $\succsim_Z$ is translation invariant on X/Z, that is, $(X/Z, \succsim_Z)$ is a preordered group.

Suppose now $\succsim_Z$ is antisymmetric, and take any $a, b \in Z$ and $x \in X$ such that $a \succsim x \succsim b$. Then, $Z = aZ \succsim_Z xZ \succsim_Z bZ = Z$ so that $xZ = Z$, that is, $x \in Z$, which proves that Z is $\succsim$-convex. Conversely, suppose Z is $\succsim$-convex, and take any x and y in X such that $xZ \succsim_Z yZ \succsim_Z xZ$. Then, $x \succsim yz_1$ and $y \succsim xz_2$ for some $z_1, z_2 \in Z$. It follows that $y \succsim xz_2 \succsim yz_1z_2$, and hence $\mathbf{1} \succsim y^{-1}xz_2 \succsim z_1z_2$ by translation invariance of $\succsim$. As both $\mathbf{1}$ and z_1z_2 belong to Z and Z is $\succsim$-convex, therefore, $y^{-1}xz_2 \in Z$. This means that $y^{-1}x \in Z$, that is, $xZ = yZ$, establishing that $\succsim_Z$ is antisymmetric.

Let $(X, \succsim)$ be a preordered topological group and Z a normal subgroup of X. We view X/Z as a topological space relative to the quotient topology (that is, a subset U of X/Z is open iff $\bigcup\{x : xZ \in U\}$ is open in X). Then, as it is well-known, X/Z is itself a topological group (which is Hausdorff iff Z is closed). In view of the preceding lemma, therefore, $(X/Z, \succsim_Z)$ is a preordered topological group in general, and it is an ordered Hausdorff topological group provided that Z is closed and $\succsim$-convex. The main result of this section is that $(X/Z,\mathrm{cl}(\succsim_Z))$ is itself an ordered Hausdorff topological group provided that X is Hausdorff and Z is compact and $\mathrm{cl}(\succsim)$-convex.

Theorem 3.3.2. *Let $(X, \succsim)$ be a preordered Hausdorff topological group and Z a compact and $\mathrm{cl}(\succsim)$-convex normal subgroup of X. Then, $(X/Z,\mathrm{cl}(\succsim_Z))$ is a partially ordered Hausdorff topological group.*

Our proof of this result is indirect, and is based on some auxiliary observations.

Lemma 3.3.3. *Let X be a Hausdorff topological group and Z a normal subgroup of X. Let $(x_\alpha)_{\alpha \in \mathbb{A}}$ be a net in X such that $x_\alpha Z \to xZ$ for some $x \in X$. Then, there is a net $(y_\beta)_{\beta \in \mathbb{B}}$ in X such that $y_\beta \to x$ and, for each $\beta \in \mathbb{B}$, $y_\beta Z = x_\alpha Z$ for some $\alpha \in \mathbb{A}$.*

Proof. Let $\pi : X \to X/Z$ be the identification map, that is, $\pi(\omega) := \omega Z$. By definition of the quotient topology, a subset U of X/Z is open iff $\pi^{-1}(U)$ is open. But, if O is an open subset of X, then $O \cdot Z$ is open in X, so, as we have $\pi^{-1}(\pi(O)) = O \cdot Z$, it must be the case that $\pi(O)$ is open in X/Z. That is, as is well known, π is an open map. It follows that $\pi(Nx)$ is an open neighborhood of xZ for every $N \in \mathcal{N}_X$. As $x_\alpha Z \to xZ$, then, for every $N \in \mathcal{N}_X$ there is an $\alpha_N \in \mathbb{A}$ such that $x_{\alpha_N} Z \in \pi(Nx)$. Consequently, for every $N \in \mathcal{N}_X$ there is a $y_N \in Nx$ such that $x_{\alpha_N} Z = y_N Z$. We now define $\mathbb{B} := \mathcal{N}_X$ to complete the proof of the lemma.

Corollary 3.3.4. *Let $(X, \succsim)$ be a preordered Hausdorff topological group and Z a compact normal subgroup of X. If $\succsim$ is continuous, so is $\succsim_Z$.*

Proof. Assume that $\succsim$ is continuous. We wish to show that $C(\succsim_Z)$ is a closed subset of X/Z (Lemma 3.1.1). To this end, take any net (x_α) in X with $x_\alpha Z \succsim_Z Z$ for each α and $x_\alpha Z \to xZ$ for some $x \in X$. We use Lemma 3.3.3 to find a net (y_β) in X, such that $y_\beta \to x$ and, for each $\beta \in \mathbb{B}$, $y_\beta Z = x_\alpha Z$, for some $\alpha \in \mathbb{A}$. Then $y_\beta Z \succsim_Z Z$ for each β, which means that there is a net (z_β) in Z such that $y_\beta \succsim z_\beta$ for each β. As Z is compact, there is a convergent subnet of (z_β); we denote this subnet also as (z_β) for simplicity. Then, as $\succsim$ is continuous, we have $x \succsim \lim z_\beta$. But as X is Hausdorff, compactness of Z implies its closedness. Thus $\lim z_\beta \in Z$, and we may conclude that $xZ \succsim_Z (\lim z_\beta)Z = Z$, that is, $xZ \in C(\succsim_Z)$.

The following is the main conclusion we wish to derive from these auxiliary resuls:

Proposition 3.3.5. *Let $(X, \succsim)$ be a preordered Hausdorff topological group and Z a compact normal subgroup of X. Then,*

$$\mathrm{cl}(\succsim_Z) = (\mathrm{cl}(\succsim))_Z.$$

Proof. By Corollary 3.3.4, $(\mathrm{cl}(\succsim))_Z$ is closed in $X/Z \times X/Z$. As $\succsim_Z \subseteq (\mathrm{cl}(\succsim))_Z$, therefore, $\mathrm{cl}(\succsim_Z) \subseteq (\mathrm{cl}(\succsim))_Z$. Conversely, take any $x, y \in X$ with $xZ\ (\mathrm{cl}(\succsim))_Z\ yZ$. Then, $x\ \mathrm{cl}(\succsim)\ yz$ for some $z \in Z$. So, we may find two nets (x_α) and (w_α) such that $x_\alpha \to x$, $w_\alpha \to yz$ and $x_\alpha \succsim w_\alpha$ for each α. But then $x_\alpha Z \to xZ$, $w_\alpha Z \to yzZ$ and $x_\alpha Z \succsim_Z w_\alpha Z$ for each α, so we have $xZ\ \mathrm{cl}(\succsim_Z)\ yzZ = yZ$. Thus: $\mathrm{cl}(\succsim_Z) \supseteq (\mathrm{cl}(\succsim))_Z$.

Proof of Theorem 3.3.2. We have already noted that $(X/Z, \succsim_Z)$ is a preordered Hausdorff topological group at the beginning of this section. By Corollary 3.1.3, therefore, $(X/Z, \mathrm{cl}(\succsim_Z))$ is a preordered Hausdorff topological group. It remains to establish the antisymmetry of $\mathrm{cl}(\succsim_Z)$. To this end, we first note that $(X, \mathrm{cl}(\succsim))$ is a preordered group by Corollary 3.1.3. By Lemma 3.3.1, therefore, $(\mathrm{cl}(\succsim))_Z$ must be antisymmetric, because Z is $\mathrm{cl}(\succsim)$-convex by hypothesis. Applying Proposition 3.3.5, then, completes the proof.

4. APPLICATIONS

4.1. Order-Dual with respect to the Closure of a Vector Order. A poset $(X, \succcurlyeq)$ is said to be a *partially ordered linear space* if X is a linear space and $\succcurlyeq$ is a vector relation on X in the sense that $\lambda x + z \succcurlyeq \lambda y + z$ holds for every $x, y, z \in X$ and $\lambda > 0$ such that $x \succcurlyeq y$. If X is a topological linear space here, we refer to $(X, \succcurlyeq)$ as a *partially ordered topological linear space*.

Let $(X, \succcurlyeq)$ be a partially ordered linear space. A subset C of X is said to be a *cone* in X if $\lambda C \subseteq C$ (that is, $\lambda x \in C$ for each $x \in C$) for all $\lambda \geq 0$, and a *convex cone* in X if C is a cone in X such that $C + C \subseteq C$. Denoting the origin of X by $\mathbf{0}$, and following the notation we used in the previous section, we define $C(\succcurlyeq) := \{x \in X : x \succcurlyeq \mathbf{0}\}$, which is a convex cone in X. In turn, we say that a linear functional $L : X \to \mathbb{R}$ is $\succcurlyeq$*-positive* if $L(C(\succcurlyeq)) \subseteq \mathbb{R}_+$, and that L is $\succcurlyeq$*-bounded* if $L(S)$ is a bounded subset of $\mathbb{R}$ whenever S is a $\succcurlyeq$-bounded subset of X. We denote the collection of all $\succcurlyeq$-positive linear functionals on X by $\mathcal{L}_+(X, \succcurlyeq)$, and that of all $\succcurlyeq$-bounded linear functionals on X by $\mathcal{L}_{\mathrm{b}}(X, \succcurlyeq)$. The former is a convex cone, and the latter a linear subspace, within the algebraic dual of X.

Let $(X, \succcurlyeq)$ be a partially ordered topological linear space such that $\mathrm{cl}(\succcurlyeq)$ is antisymmetric. (This is the case, for instance, when $\succcurlyeq$ is normal.) Then, $(X,\mathrm{cl}(\succcurlyeq))$ is also a partially ordered topological linear space, and it is natural to inquire under what conditions the order-duals of $(X, \succcurlyeq)$ and $(X,\mathrm{cl}(\succcurlyeq))$ would be identical. The following result shows that this indeed happens under fairly standard conditions.

Theorem 4.1.1. *Let $(X, \succcurlyeq)$ be a normally-ordered locally convex Hausdorff topological linear space. If $\mathrm{int}(C(\succcurlyeq)) \neq \emptyset$, then $\mathcal{L}_{\mathrm{b}}(X, \succcurlyeq) = \mathcal{L}_{\mathrm{b}}(X,\mathrm{cl}(\succcurlyeq))$.*

Proof. Evidently, $x \ \mathrm{cl}(\succcurlyeq)\ y$ iff $\lambda x \ \mathrm{cl}(\succcurlyeq)\ \lambda y$, for all $x, y \in X$ and $\lambda > 0$. We can therefore invoke Proposition 3.2.1 to conclude that $(X,\mathrm{cl}(\succcurlyeq))$ is a normally ordered topological linear space. We now recall that the order-dual of a normally ordered locally convex topological linear space whose positive cone has nonempty interior is the span of the convex cone of all positive linear functionals on that space. (See, for instance, [**1**], Corollary 2.27.) It follows that the order-duals of $(X, \succcurlyeq)$ and $(X,\mathrm{cl}(\succcurlyeq))$ equal the spans of $\mathcal{L}_+(X, \succcurlyeq)$ and $\mathcal{L}_+(X,\mathrm{cl}(\succcurlyeq))$, respectively.

As it is obvious that $\mathcal{L}_+(X,\mathrm{cl}(\succcurlyeq))$ is contained in $\mathcal{L}_+(X, \succcurlyeq)$, it remains to show that $\mathcal{L}_+(X, \succcurlyeq) \subseteq \mathcal{L}_+(X,\mathrm{cl}(\succcurlyeq))$. To this end, pick any $\succcurlyeq$-positive linear functional L on X. We recall that a positive linear functional on a partially ordered topological linear space is continuous, provided that the positive cone of that space has nonempty interior. (This is a special case of the Nachbin-Namioka-Schaefer Theorem; see Proposition 2.16 of [**18**].) Thus, under the assumption $\mathrm{int}(C(\succcurlyeq)) \neq \emptyset$, our L must be continuous. Then, part (c) of Lemma 3.1.1 implies

$$L(C(\mathrm{cl}(\succcurlyeq))) = L(\mathrm{cl}(C(\succcurlyeq))) \subseteq \mathrm{cl}(L(C(\succcurlyeq))) \subseteq \mathbb{R}_+,$$

that is, $L \in \mathcal{L}_+(X,\mathrm{cl}(\succcurlyeq))$. It follows that $\mathcal{L}_+(X, \succcurlyeq) = \mathcal{L}_+(X,\mathrm{cl}(\succcurlyeq))$, and our proof is complete.

4.2. Continuous Extension of Partial Orders on Groups. A preorder $\trianglerighteq$ on a nonempty set X is said to *extend* another preorder $\succsim$ on X if the symmetric and asymmetric parts of $\succsim$ are contained in those of $\trianglerighteq$, respectively. The problem of extending a given preorder that satisfies a given property to a total preorder which satisfies the same property is a major topic of interest in order theory and its applications. In particular, in topological order theory, an important problem is to determine when a given continuous preorder on a topological space can be extended to a continuous total preorder; see, for instance, [**9**] and [**2**]. We now provide an extension theorem of this flavor in the context of topological groups.

Theorem 4.2.1. *Let $(X, \succsim)$ be a preordered commutative topological group. If $\succsim$ is open-continuous, there exists a continuous and translation invariant total preorder on X that extends $\succsim$.*

Proof. In [**17**] it is proved that every translation invariant preorder on a commutative group can be extended to a translation invariant total preorder on that group. Therefore, let $\succsim^*$ be any translation invariant total preorder that extends $\succsim$. By Corollary 3.1.3, $\mathrm{cl}(\succsim^*)$ is a continuous and translation invariant preorder on X. As $\succsim^*$ is total, so is $\mathrm{cl}(\succsim^*)$. Finally, take any $x, y \in X$ with $x \succ y$. Then, by open-continuity of $\succsim$, there are open neighborhoods U and V of x and y, respectively, such that $u \succ v$ for every $(u, v) \in U \times V$. As $\succsim^*$ extends $\succsim$, we also have $u \succ^* v$ for every $(u, v) \in U \times V$, where $\succ^*$ is the asymmetric part of $\succsim^*$. It follows readily from this fact that $y \ \mathrm{cl}(\succsim^*) \ x$ cannot hold. As $\mathrm{cl}(\succsim^*)$ is total, therefore, (x, y) is contained in the asymmetric part of $\mathrm{cl}(\succsim^*)$. Conclusion: $\mathrm{cl}(\succsim^*)$ extends $\succsim$.

There are a few interesting open problems that emanate from these results. First, we do not know, but conjecture that it is not true, if one can replace the open-continuity condition in Theorem 4.2.1 with continuity. Second, it is not clear when one can ensure that the extending preorder found in Theorem 4.2.1 is either a partial or a lattice order.

4.3. Numerical Representation of Preorders. Let $(X, \succsim)$ be a preordered set. Recall that a function $f : X \to \mathbb{R}$ is said to be *$\succsim$-increasing* if $f(x) \geq f(y)$ for every $x, y \in X$ with $x \succsim y$, and *strictly $\succsim$-increasing* if it is $\succsim$-increasing and $f(x) > f(y)$ for every $x, y \in X$ with $x \succ y$. We say that $\succsim$ admits a *numerical representation* if there exists a strictly $\succsim$-increasing real map on X.

In decision theory, where $(X, \succsim)$ is a preordered topological space, a strictly $\succsim$-increasing and continuous real map on X is called a *Richter-Peleg utility function*, and if there exists such a function, $\succsim$ is said to admit a *Richter-Peleg representation*. (See [**3**] and [**7**] for outlines of the theory of Richter-Peleg utility representation.) In this terminology, *Levin's Theorem* – see [**12**] – says that every continuous preorder on a locally compact and second countable Hausdorff topological space X admits a Richter-Peleg representation. The following result gives an instance in which this result extends to the case of open-continuous preorders.

Theorem 4.3.1. *Let X be a locally compact and second countable Hausdorff topological group. Then, every open-continuous and translation invariant preorder $\succsim$ on X admits a Richter-Peleg representation.*

Proof. Let $\succsim$ be an open-continuous and translation invariant preorder $\succsim$ on X. By Proposition 3.1.2, $\mathrm{cl}(\succsim)$ is a continuous preorder on X, so by Levin's Theorem, there is a continuous and strictly $\mathrm{cl}(\succsim)$-increasing real map on X. As $\succsim$ is open-continuous, this map is strictly $\succsim$-increasing as well.

Corollary 4.3.2. *Every open-continuous and translation invariant preorder on a compact Hausdorff topological group X admits a Richter-Peleg representation.*

Remark 4.3.3. (a) The classical Birkhoff-Kakutani Theorem says that every first (and hence second) countable topological group is metrizable. But a locally compact and second countable metrizable topological space is homeomorphic to a locally compact and separable metric space. It is thus without loss of generality to take X in Theorem 4.3.1 as a locally compact and separably metrizable topological group.

(b) We can generalize Levin's Theorem, and hence Theorem 4.3.1, by replacing the second countability property with σ-compactness. The proof, which is routine, is sketched in [**7**].

(c) Theorem 4.3.1 complements the main theorem of [**4**] which states that every open-continuous and order-separable partial order on a locally compact commutative Hausdorff topological group admits a Richter-Peleg representation. In effect, Theorem 4.3.1 shows that if the topological group at hand has a countable basis (or is σ-compact), then in the said result we can omit the requirements of commutativity, antisymmetry and order-separability. In particular, for compact Hausdorff topological groups, Corollary 4.3.2 is more general than that theorem.

4.4. Existence of Extremal Elements Relative to a Partial Order. Let $(X, \succsim)$ be a preordered set, and recall that an element x of a subset S of X is said to be $\succsim$*-maximal* in S if $y \succ x$ does not hold for any $y \in S$. (Here $\succ$ is the asymmetric part of $\succsim$.) In turn, x is said to be $\succsim$*-minimal* in S if it is $\precsim$*-maximal* in S. We denote the set of all $\succsim$-maximal and $\succsim$-minimal elements in S by $\mathbf{MAX}(S, \succsim)$ and $\mathbf{MIN}(S, \succsim)$, respectively.

The following result provides sufficient conditions for the existence of extrema in a set with respect to a given translation invariant partial order on a topological group.

Theorem 4.4.1. *Let $(X, \succcurlyeq)$ be a normally ordered Hausdorff topological group, and S a nonempty compact subset of X. Then,* $\mathbf{MAX}(S, \succcurlyeq) \neq \emptyset$ and $\mathbf{MIN}(S, \succcurlyeq) \neq \emptyset$.

Proof. By Proposition 3.2.1, $\mathrm{cl}(\succcurlyeq)$ is a partial order on X, and due to the antisymmetry of $\mathrm{cl}(\succcurlyeq)$, an element x of S is $\mathrm{cl}(\succcurlyeq)$-maximal in S iff $y\ \mathrm{cl}(\succcurlyeq)\ x$ fails for any $y \in S \backslash \{x\}$. It follows that $\mathbf{MAX}(S,\mathrm{cl}(\succcurlyeq)) \subseteq \mathbf{MAX}(S, \succcurlyeq)$. But a classical theorem of order theory says that every compact topological space admits a maximal element relative to a continuous partial order on that space.[6] We thus have $\mathbf{MAX}(S,\mathrm{cl}(\succcurlyeq)) \neq \emptyset$, and our first assertion is proved. The second assertion follows from applying what we have just established in the context of the poset $(X, \preccurlyeq)$.

4.5. The Topological/Transitive Closure Problem. In his famous dissertation, [**10**] showed that a subset of a topological space can generate at most 14 sets upon taking topological closures and complements. A similar result is proved by [**8**] where it is shown that a binary relation on a (nonempty) set can generate at most 10 binary relations under the operations of transitive closure and complementation. (The *transitive closure* of a binary relation R on nonempty set X, denoted by $\mathrm{tran}(R)$, is the smallest transitive relation on X that contains R, and is given by $\mathrm{tran}(R) := R \cup R^2 \cup \cdots$.) We are then led to the following *topological/transitive closure problem*: How many distinct binary relations may be obtained by taking topological and transitive closures of a binary relation defined on a topological space? [**13**] have recently shown that there is no finite upper bound to this effect. (In fact, a binary relation may well generate infinitely many distinct binary relations by applying these two closure operators.) However, it is still of interest to find out if a finite bound can be obtained by imposing suitable conditions on the

[6] This result goes back to [**21**], but see also [**22**].

binary relation and the topological space under consideration. The following is a result of this form.

Theorem 4.5.1. *Let R be a reflexive translation invariant binary relation on a topological group X. Then, at most 4 relations can be generated from R by taking topological and transitive closures.* (*This is the best possible bound.*)

Proof. It is plain that $\text{tran}(R)$ is a preorder on X. This relation is also translation invariant. To see this, take any x and y in X, and suppose there is a finite subset $\{a_1, ..., a_n\}$ of X such that $x = a_1 \; R \cdots R \; a_n = y$. As $a_i \; R \; a_{i+1}$ implies $a_i\omega \; R \; a_{i+1}\omega$ for each $i = 1, ..., n-1$, we see that $x\omega = a_1\omega \; R \cdots R \; a_n\omega = y\omega$, and hence $x\omega \; \text{tran}(R) \; y\omega$, for every $\omega \in X$. One can show analogously that $\omega x \; \text{tran}(R) \; \omega y$ for every $\omega \in X$. Consequently, we may apply Proposition 3.1.2 to conclude that $\text{cl}(\text{tran}(R))$ is a preorder. As this relation obviously contains $\text{cl}(R)$, therefore, $\text{tran}(\text{cl}(R))) \subseteq \text{cl}(\text{tran}(R))$, and hence, $\text{cl}(\text{tran}(\text{cl}(R))) \subseteq \text{cl}(\text{tran}(R))$. As the converse containment is trivially true, then, we may conclude that $\text{cl}(\text{tran}(\text{cl}(R))) = \text{cl}(\text{tran}(R))$. Consequently, any binary relation that can be generated from R under the operations of transitive closure and topological closure must be included in the set $\{\text{cl}(R), \text{tran}(R), \text{tran}(\text{cl}(R)), \text{cl}(\text{tran}(R))\}$.

It remains to prove that 4 is the best possible upper bound in the context of Theorem 4.5.1. In what follows, we write $\hat{x}$ for the 3-vector (x_1, x_2, x_3) and $\tilde{x}$ for (x_4, x_5) for any given 5-vector $x := (x_1, ..., x_5)$. Let us pick a closed cone C in $\mathbb{R}^3$ such that $\text{conv}(C)$ is not closed (cf. [**11**]), and define the binary relation S on $\mathbb{R}^3$ by $x \; S \; y$ iff $x - y \in C$. Next, we define the binary relation R on $\mathbb{R}^5$ by $x \; R \; y$ iff $\hat{x} \; S \; \hat{y}$ and either $\tilde{x} \gg \tilde{y}$ or $\tilde{x} = \tilde{y}$.[7] This is a vector relation, and for any x and y in $\mathbb{R}^5$, we have

a. $x \; \text{cl}(R) \; y$ iff $\hat{x} \; S \; \hat{y}$ and $\tilde{x} \geq \tilde{y}$;
b. $x \; \text{tran}(\text{cl}(R)) \; y$ iff $\hat{x} \; \text{tran}(S) \; \hat{y}$ and $\tilde{x} \geq \tilde{y}$;
c. $x \; \text{tran}(R) \; y$ iff $\hat{x} \; \text{tran}(S) \; \hat{y}$ and either $\tilde{x} \gg \tilde{y}$ or $\tilde{x} = \tilde{y}$;
d. $x \; \text{cl}(\text{tran}(R)) \; y$ iff $\hat{x} \; \text{cl}(\text{tran}(S)) \; \hat{y}$ and $\tilde{x} \geq \tilde{y}$;

As all four of these binary relations are distinct, we are done.

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[7] Here $\tilde{x} \gg \tilde{y}$ means $x_4 > y_4$ and $x_5 > y_5$.

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Department of Economics, NYU
E-mail address: `efe.ok@nyu.edu`

Department of Economics, Universidade de Brasilia
E-mail address: `riella@unb.br`